Pergamon

Int. J. Heat Mass Transfer. Vol. 38, No. 15, pp. 2915–2919, 1995

0017-9310/95 $9.50+0.00

0017-9310(95)00035-6

TECHNICAL NOTES

Analytical solution for transient heat conduction in hollow cylinders containing well-stirred fluid with uniform heat sink

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(*Received* 27 *September* 1994 *and in final form* 12 *December* 1994)

INTRODUCTION

Heat conduction is the most important mechanism for thermal energy transportation in solid objects. The successful progress in applied and theoretical mathematics about the initial-boundary value problems of a general parabolic system [1] has solidified the fundamental knowledge in finding solutions for specified engineering problems. In early times, when the computational capability was not so powerful as today, a large amount of effort had to be devoted to finding analytical solutions in applicable forms following very strict mathematical disciplines. Nowadays, superior computer capacities and contemporary numerical techniques have become available for numerically solving heat conduction problems described by parabolic equations, while the classical way has largely been ignored. Theoretical research in the heat transfer field, regarding both conduction and convection, is mainly by numerical simulations. However, the classical way for finding analytical solutions is obviously still important for verifying numerical solutions which are less strict than formerly. A numerical solution must be shown to be acceptable based on the combination of the following three criteria [2]:

1. estimates of computing error bounds;
2. comparisons with *analytical solutions*;
3. substituting the results into the original equations.

The experimental data are useful to assess the mathematical models, but are never sufficient to verify the numerical solutions of the established mathematical models. Comparisons between the numerical calculations and the experimental data fail to reveal the compensation of modelling deficiencies through computing errors or unconscious approximations in establishing applicable numerical schemes. Moreover, analytical solutions for some specified problems are also essential for the development of efficient applied numerical simulation tools.

The present paper discusses a practical method to derive a closed form of the exact solution for transient heat conduction in hollow cylinders containing well-stirred fluid with uniform heat sink (or source). The physical problem is depicted graphically by Fig. 1 and mathematically by

$$\frac{1}{a}\frac{\partial T_w}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_w}{\partial r}\right)$$

$$\lambda_w \frac{\partial T_w}{\partial r}\bigg|_{r=r_0} = 0$$

$$\lambda_w \frac{\partial T_w}{\partial r}\bigg|_{r=r_i} = h_w(T_w(r_i,t) - T_f(t))$$

$$M_f C_{v,f}\frac{dT_f}{dt} = h_w A(T_w(r_i,t) - T_f(t)) - q_v V_f \qquad (1)$$

with Initial Conditions (ICs):

$$T_w(r,0) = T_f(0) = T_0.$$

In the above equation, $C_{v,f}$ is used without losing generality, which implies a process with a constant volume. The above equation is still valid in a process at constant pressure, but $C_{p,f}$ instead of $C_{v,f}$ should be employed.

Although this is an old heat conduction problem, in which the solid walls are in contact with the well-stirred fluid, the only solutions that the author could find in open literature are always addressed using the assumption (e.g. [3]):

$$T_f(t) = T_w(r_i,t), \quad t \geqslant 0. \qquad (2)$$

The early achievements related to heat conduction in solids were largely encompassed in the famous textbook written by Carslaw and Jaeger [4]. Van Sant [5] later also assembled a collection of conduction heat transfer solutions that he had found in numerous publications. However, the solution for the more general case shown in equation (1) has not yet been found. The present study derives a solution for such a specific problem by applying the Laplace transform method.

SOLUTION PROCEDURE

For simplicity in processing solutions with the Laplace transform method, equation (1) is modified into the fol-

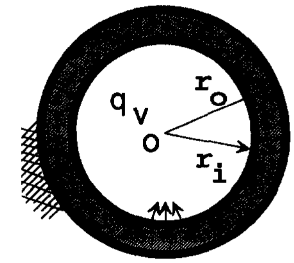

Fig. 1. A typical hollow cylinder containing well-stirred fluid with uniform heat sink.

NOMENCLATURE

A	heat transfer area [m^2]
a	thermal diffusivity [$m^2\ s^{-1}$]
C_v	specific heat at constant volume [$J\ kg^{-1}\ K^{-1}$]
h_w	heat transfer coefficient [$W\ m^{-2}\ K^{-1}$]
I_0, I_1	the zero and first-order modified Bessel functions
J_0, J_1	the zero and first-order Bessel functions
K_0, K_1	the zero and first-order second-type modified Bessel functions
M_f	mass of the contained fluid [kg]
$\dot{q}_v$	volumetric heat sink [$W\ m^{-3}$]
r	radial position in the cylindrical wall, $r_i \leqslant r \leqslant r_0$ [m]
s	Laplace variable [s^{-1}]
T	temperature [K]
t	time [s]
V_f	volume of the contained fluid [m^3]
Y_0, Y_1	the zero and first-order second-type Bessel functions.

Greek symbols

λ_w	heat conductivity of the wall [$W\ m^{-1}\ K^{-1}$]
ρ	density [$kg\ m^{-3}$].

Subscripts and superscripts

f	fluid
$\hat{f}$	Laplace-transform of function f
w	wall
0	initial value.

lowing standard *zero* initial-boundary value form as:

$$\frac{1}{a}\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$\lambda_w \left.\frac{\partial u}{\partial r}\right|_{r=r_0} = 0$$

$$\lambda_w \left.\frac{\partial u}{\partial r}\right|_{r=r_i} = h_w(u(r_i,t)-w(t))$$

$$\frac{dw}{dt} = \frac{h_w A}{M_f C_{v,f}}[u(r_i,t)-w(t)] + \frac{q_v V_f}{M_f C_{v,f}} \tag{3}$$

with ICs:

$$u(r,0) = w(0) = 0$$

where u, w, the excess temperatures for both the cylindrical wall and the contained fluid, are defined as below, respectively:

$$u(r,t) = T_0 - T_w(r,t)$$

and

$$w(t) = T_0 - T_f(t). \tag{4}$$

The Laplace transform for w in equation (3) yields

$$\hat{w}(s) = \left(s+\frac{1}{\tau_f}\right)^{-1}\left(\frac{q_v V_f}{M_f C_{v,f}}\frac{1}{s}+\frac{1}{\tau_f}\hat{u}(r_i,s)\right) \tag{5}$$

and the Laplace transform for u in equation (3) leads to:

$$\frac{\partial^2 \hat{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \hat{u}}{\partial r}-\frac{s}{a}\hat{u} = 0 \tag{6}$$

which is subject to the solution of $\hat{u}$ as:

$$\hat{u}(r,s) = C_1 I_0\left(\sqrt{\frac{s}{a}}r\right)+C_2 K_0\left(\sqrt{\frac{s}{a}}r\right). \tag{7}$$

The adiabatic boundary condition at $r = r_0$ in equation (3) gives

$$\hat{u}(r,s) = \frac{C_2}{I_1\left(\sqrt{\frac{s}{a}}r_0\right)}\left[K_1\left(\sqrt{\frac{s}{a}}r_0\right)I_0\left(\sqrt{\frac{s}{a}}r\right) + I_1\left(\sqrt{\frac{s}{a}}r_0\right)K_0\left(\sqrt{\frac{s}{a}}r\right)\right]. \tag{8}$$

By eliminating $\hat{w}$ and $\hat{u}(r_i, s)$ with the combination of equations (5), (8) and the boundary condition at $r = r_i$ in equation (3) in terms of Laplace transforms, one obtains:

$$C_2 = \frac{(q_v V_f/A)(1/(s+\tau_f^{-1})-1/s)I_1(\sqrt{s/a}\,r_0)}{\Pi(s)}$$

$$\tau_f = \frac{M_f C_{v,f}}{h_w A}$$

$$\Pi(s) = \lambda_w\sqrt{\frac{s}{a}}\left[K_1\left(\sqrt{\frac{s}{a}}r_0\right)I_1\left(\sqrt{\frac{s}{a}}r_1\right) - I_1\left(\sqrt{\frac{s}{a}}r_0\right)K_1\left(\sqrt{\frac{s}{a}}r_i\right)\right] - \frac{h_w s}{s+\tau_f^{-1}}\left[K_1\left(\sqrt{\frac{s}{a}}r_0\right)I_0\left(\sqrt{\frac{s}{a}}r_i\right) + I_1\left(\sqrt{\frac{s}{a}}r_0\right)K_0\left(\sqrt{\frac{s}{a}}r_i\right)\right]. \tag{9}$$

By substituting equation (9) into equation (8) and rearranging terms in order to process an inverse Laplace transform, one has

$$\hat{u}(r,s) = \hat{g}_1(s)\cdot\hat{g}_2(r,s)$$

$$\hat{g}_1(s) = -\left(\frac{q_v V_f}{A\tau_f}\right)\frac{1}{s^2}$$

$$\hat{g}_2(r,s) = \frac{\sqrt{s/a}[K_1(\sqrt{s/a}r_0)I_0(\sqrt{s/a}r) + I_1(\sqrt{s/a}r_0)K_0(\sqrt{s/a}r)]}{(\lambda_w/a)(s+\tau_f^{-1})\psi_1(s) - h_w\sqrt{s/a}\psi_2(s)} \tag{10}$$

where

$$\psi_1(s) = K_1\left(\sqrt{\frac{s}{a}}r_0\right)I_1\left(\sqrt{\frac{s}{a}}r_i\right) - I_1\left(\sqrt{\frac{s}{a}}r_0\right)K_1\left(\sqrt{\frac{s}{a}}r_i\right)$$

$$\psi_2(s) = K_1\left(\sqrt{\frac{s}{a}}r_0\right)I_0\left(\sqrt{\frac{s}{a}}r_i\right)$$

$$-I_1\left(\sqrt{\frac{s}{a}}r_0\right)K_0\left(\sqrt{\frac{s}{a}}r_i\right). \quad (11)$$

According to the rule of Laplace transforms (e.g. [6]), one has:

If $\mathbf{L}(f(t)) = \hat{g}_1(s)\cdot\hat{g}_2(s)$ and the inverse Laplace transform $g_1(t)$ and $g_2(t)$ are known as $g_1(t) = \mathbf{L}^{-1}(\hat{g}_1(s))$ and $g_2(t) = \mathbf{L}^{-1}(\hat{g}_2(s))$, then the inverse Laplace transform $f(t)$ is defined by:

$$f(t) = \int_0^t g_1(t-\tau)g_2(\tau)\,\mathrm{d}\tau. \quad (12)$$

In our special case $\hat{g}_1(s)$ and $\hat{g}_2(r,s)$ are shown in equation (10). The inverse Laplace transforms for both $\hat{g}_1(s)$ and $\hat{g}_2(r,s)$ are found to be:

$$g_1(t) = -\left(\frac{q_v V_f}{A\tau_f}\right)t$$

$$g_2(r,t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp(st)\cdot\hat{g}_2(r,s)\,\mathrm{d}s = \sum_i \mathrm{Res}_i(\exp(st)\hat{g}_2(r,s)). \quad (13)$$

Equation (10) for $\hat{g}_2(r,s)$ can be simplified as follows with the variable definition $q = \sqrt{s/a}$:

$$\hat{g}_2(r,s) = \hat{g}_2(r,q) = \frac{q[K_1(qr_0)I_0(qr)+I_1(qr_0)K_0(qr)]}{(\lambda_w/a)(aq^2+\tau_f^{-1})\psi_1(q)-h_w q\psi_2(q)}. \quad (14)$$

To find the residuals of the integrant depicted in equation (13), one needs to search all pole singulars of the above equation, corresponding to all zero points of its determination. Therefore, these singulars can be specified by the roots of the following transcendent characteristic equation:

$$\Delta(q) \equiv \frac{\lambda_w}{a}\left(aq^2+\frac{1}{\tau_f}\right)\psi_1(q) - h_w q\psi_2(q) = 0. \quad (15)$$

The roots of the above equation are real and can be determined by transforming into the real domain with

$$s = -a\beta^2 \quad \text{or} \quad q = i\beta. \quad (16)$$

Then by using Wronskian recurrence relations [7], one obtains

$$\Delta(i\beta) = \frac{\lambda_w\pi}{2a}\left(\frac{1}{\tau_f}-a\beta^2\right)(Y_1(\beta r_i)J_1(\beta r_0) - J_1(\beta r_i)Y_1(\beta r_0)) - \frac{\pi}{2}h_w\beta(Y_0(\beta r_i)J_1(\beta r_0) - J_0(\beta r_i)Y_1(\beta r_0)) = 0. \quad (17)$$

The above equation leads to

$$\frac{\lambda_w}{ah_w}\left(\frac{1}{\tau_f}+aq^2\right) = q\frac{\psi_2(q)}{\psi_1(q)} = \frac{\lambda_w}{ah_w}\left(\frac{1}{\tau_f}-a\beta^2\right) = \frac{\beta(Y_0(\beta r_i)J_1(\beta r_0)-J_0(\beta r_i)Y_1(\beta r_0))}{(Y_1(\beta r_i)J_1(\beta r_0)-J_1(\beta r_i)Y_1(\beta r_0))}. \quad (18)$$

According to the complex analysis theory and Laurant series, if $q_n = i\beta_n$ are simple singulars, which should holds true in our special case, then

$$\mathrm{Res}_n(\exp(s_n t)\hat{g}_2(r,s_n)) = \frac{\lim_{s\to s_n}\exp(aq^2t)q(K_1(qr_0)I_0(qr)+I_1(qr_0)K_0(qr))}{\lim_{s\to s_n}\frac{\mathrm{d}\Delta}{\mathrm{d}s}}. \quad (19)$$

The definition of q implies:

$$\frac{\mathrm{d}\Delta}{\mathrm{d}s} = \frac{\mathrm{d}\Delta}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{1}{2aq}\frac{\mathrm{d}\Delta}{\mathrm{d}q}. \quad (20)$$

Again, by applying Wronskian recurrence relations [7] to transform Bessel functions from the complex domain to the real domain, one obtains:

$$\begin{aligned}F(\beta) \equiv \frac{\mathrm{d}\Delta}{\mathrm{d}s} &= \frac{\lambda_w\pi}{2a}(Y_1(\beta r_i)J_1(\beta r_0) - J_1(\beta r_i)Y_1(\beta r_0)) \\ &- \frac{h_w\pi}{2a\beta}(Y_1(\beta r_0)J_0(\beta r_i) - J_1(\beta r_0)Y_0(\beta r_i)) \\ &+ \frac{h_w r_0\pi}{4a}\Bigg[J_0(\beta r_0)Y_0(\beta r_i) - J_0(\beta r_i)Y_0(\beta r_0) \\ &+ \frac{[Y_1(\beta r_0)J_0(\beta r_i) - Y_0(\beta r_i)J_1(\beta r_0)][Y_1(\beta r_i)J_0(\beta r_0) - J_1(\beta r_i)Y_0(\beta r_0)]}{(Y_1(\beta r_i)J_1(\beta r_0) - J_1(\beta r_i)Y_1(\beta r_0))}\Bigg] \\ &- \frac{h_w r_i\pi}{4a}\Bigg(Y_1(\beta r_i)J_1(\beta r_0) - J_1(\beta r_i)Y_1(\beta r_0) \\ &+ \frac{[Y_1(\beta r_0)J_0(\beta r_i) - Y_0(\beta r_i)J_1(\beta r_0)]^2}{[Y_1(\beta r_i)J_1(\beta r_0) - J_1(\beta r_i)Y_1(\beta r_0)]}\Bigg) \quad (21)\end{aligned}$$

and

$$q[K_1(qr_0)I_0(qr)+I_1(qr_0)K_0(qr)] = -\frac{\beta\pi}{2}[Y_1(\beta r_0)J_0(\beta r) - J_1(\beta r_0)Y_0(\beta r)]. \quad (22)$$

Finally, by substituting the above equation into equation (13), one arrives at

$$g_2(r,t) = \sum_n \mathrm{Res}_n(\exp(s_n t)\hat{g}_2(r,s_n)) = \sum_n \frac{\beta_n\pi}{2F(\beta_n)}(J_1(\beta_n r_0)Y_0(\beta_n r) - Y_1(\beta_n r_0)J_0(\beta_n r))\exp(-a\beta_n^2 t). \quad (23)$$

According to the rule of the inverse Laplace transform, we obtain

$$u(r,t) = \int_0^t g_1(t-\tau)g_2(r,\tau)\,\mathrm{d}\tau = \sum_n\left(\frac{q_v V_f}{A\tau_f}\right)\left(\frac{\beta_n\pi}{2F(\beta_n)}\right) \times (J_0(\beta_n r)Y_1(\beta_n r_0) - Y_0(\beta_n r)J_1(\beta_n r_0)) \cdot\left(\frac{t}{a\beta_n^2} - \frac{1-\exp(-a\beta_n^2 t)}{a^2\beta_n^4}\right). \quad (24)$$

By converting $u(r,t)$ back to $T_w(r,t)$ with equation (4) and writing it into a dimensionless form, one has

$$\Theta_w(r,t) \equiv \frac{T_w(r,t)}{T_0} = 1 - \frac{u(r,t)}{T_0} = 1 - \sum_n\left(\frac{q_v V_f}{AT_0\tau_f}\right)\left(\frac{\beta_n\pi}{2F(\beta_n)}\right) \times (J_0(\beta_n r)Y_1(\beta_n r_0) - Y_0(\beta_n r)J_1(\beta_n r_0)) \cdot\left(\frac{t}{a\beta_n^2} - \frac{1-\exp(-a\beta_n^2 t)}{a^2\beta_n^4}\right). \quad (25)$$

By further integrating the equation for $w(t)$ in equation (3) and coupling with the solution for $u(r,t)$, one gets

$$w(t) = \left(\frac{q_v V_f}{Ah_w}\right)\left[1-\exp\left(-\frac{t}{\tau_f}\right)\right]$$
$$+\sum_n \left(\frac{q_v V_f}{A\tau_f}\right)\left(\frac{\beta_n \pi}{2F(\beta_n)}\right)$$
$$\times (J_0(\beta_n r_i)Y_1(\beta_n r_0) - Y_0(\beta_n r_i)J_1(\beta_n r_0))$$
$$\times \left[\frac{t}{a\beta_n^2} + \left(\tau_f + \frac{1}{a\beta_n^2}\right)\frac{\exp(-t/\tau_f)-1}{a\beta_n^2}\right.$$
$$\left. - \frac{\exp(-a\beta_n^2 t) - \exp(-t/\tau_f)}{a^2\beta_n^4(a\beta_n^2\tau_f - 1)}\right]. \quad (26)$$

Similarly, the dimensionless form of the fluid temperature can be written as,

$$\Theta_f(t) \equiv \frac{T_f(t)}{T_0} = 1 - \frac{w(t)}{T_0} = 1$$
$$-\left(\frac{q_v V_f}{Ah_w T_0}\right)\left[1-\exp\left(-\frac{t}{\tau_f}\right)\right]$$
$$-\sum_n \left(\frac{q_v V_f}{AT_0\tau_f}\right)\left(\frac{\beta_n \pi}{2F(\beta_n)}\right)$$
$$\times (J_0(\beta_n r_i)Y_1(\beta_n r_0) - Y_0(\beta_n r_i)J_1(\beta_n r_0))$$
$$\times \left[\frac{t}{a\beta_n^2} + \left(\tau_f + \frac{1}{a\beta_n^2}\right)\frac{\exp(-t/\tau_f)-1}{a\beta_n^2}\right.$$
$$\left. - \frac{\exp(-a\beta_n^2 t) - \exp(-t/\tau_f)}{a^2\beta_n^4(a\beta_n^2\tau_f - 1)}\right]. \quad (27)$$

Equations (25) and (27) are the solutions of the problem discussed here for the wall and the fluid respectively. Because the characteristics of all types of Bessel functions have been elaborately investigated and all functions can be precisely evaluated from the existing mathematical establishment (e.g. [7]), the applications of the solutions shown in equations (25) and (27) can be easily manipulated.

CONCLUSION

A closed form of the analytical solution for transient heat conduction in hollow cylinders containing well-stirred fluid with uniform heat sink is found using Laplace transform methodology. The solution is also valid for the case of containing a well-stirred fluid with uniform heat source by changing q_v into $-q_v$. The practices used in the present study are also applicables with minor modifications, to similar problems with more general boundary conditions. It is expected that the present study will be useful for enlarging the fundamental mathematical knowledge base of transient heat conduction, and for possible applications in some engineering fields, such as micro-heat pipes, liquid metal cooling systems and vessel protection, etc.

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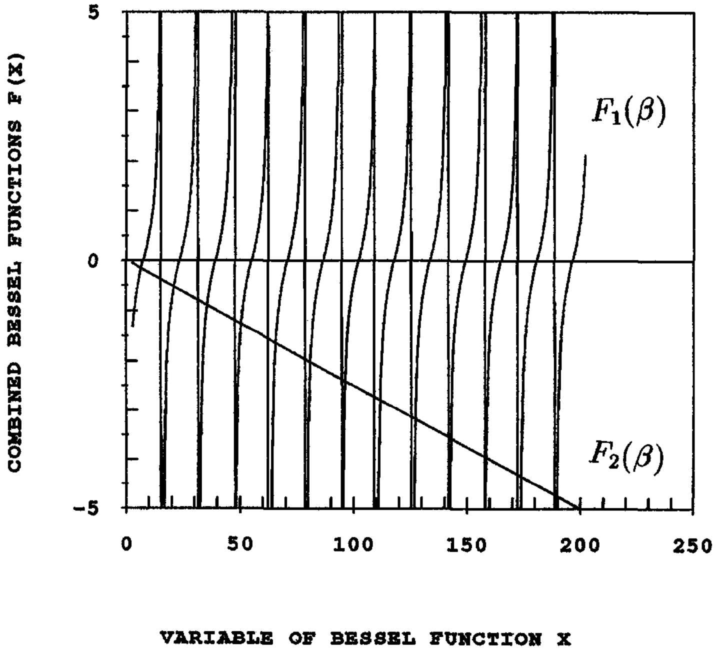

Fig. B1. Calculation for β_n.

APPENDIX A: WANSKIAN RECURRENCE RELATIONS

$$zI'_v(z)+vI_v(z) = zI_{v-1}(z),$$
$$zI'_v(z)-vI_v(z) = zI_{v+1}(z)$$
$$zK'_v(z)+vK_v(z) = -zK_{v-1}(z),$$
$$zK'_v(z)-vK_v(z) = -zK_{v+1}(z)$$
$$zJ'_v(z)+vJ_v(z) = zJ_{v-1}(z),$$
$$zJ'_v(z)-vJ_v(z) = -zJ_{v+1}(z)$$
$$I'_0(z) = I_1(z), \quad K'_0(z) = -K_1(z)$$
$$J'_0(z) = -J_1(z), \quad Y'_0(z) = -Y_1(z)$$
$$J_v(z)Y'_v(z)-Y_v(z)J'_v(z) = \frac{2}{\pi z}$$
$$I_v(z)K'_v(z)-K_v(z)I'_v(z) = -\frac{1}{z}$$
$$I_v(z)K_{v+1}(z)+K_v(z)I_{v+1}(z) = \frac{1}{z}$$
$$J_v(z\,e^{m\pi i}) = e^{mv\pi i}J_v(z),$$
$$I_v(z\,e^{\pm 1/2\pi i}) = e^{\pm 1/2v\pi i}J_v(z)$$
$$K_v(z\,e^{\pm 1/2\pi i}) = \pm\tfrac{1}{2}\pi i\,e^{\mp 1/2v\pi i}[-J_v(z)\pm iY_v(z)]$$
$$Y_v(z\,e^{m\pi i}) = e^{-mv\pi i}Y_v(z)+2i\sin(mv\pi)\cot(v\pi)J_v(z)$$

APPENDIX B: EXAMPLES TO CALCULATE ROOTS β_n OF EQUATION (17)

$$F_1(\beta) = \frac{Y_0(\beta r_i)J_1(\beta r_0)-J_0(\beta r_i)Y_1(\beta r_0)}{Y_1(\beta r_i)J_1(\beta r_0)-J_1(\beta r_i)Y_1(\beta r_0)}$$
$$F_2(\beta) = \frac{\lambda_w}{ah_w}\left(\frac{1}{\tau_f\beta}-a\beta\right)$$

If $F_1(\beta) = F_2(\beta)$, then $\beta = \beta_n$.

Pergamon

Int. J. Heat Mass Transfer. Vol. 38, No. 15, pp. 2919–2922, 1995

0017–9310/95 $9.50+0.00

0017–9310(95)00007–0

Heat wave phenomena in IC chips

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(*Received* 12 *April* 1994 *and in final form* 5 *December* 1994)

INTRODUCTION

Thermal management is becoming a predominant consideration in the design of IC chips and their packaging. The electrical behavior of devices and their reliability are strongly dependent both on the temperature of chip and temperature difference among the components. Many researchers pay their attention to the failure resulting from an overhigh chip temperature, which is always associated with irreversible mechanical fracture as well as loss of electrical functions. In contrast our efforts have concentrated on the analysis of the thermal failure arising from temperature difference among the components related to critical electrical paths. For a high-speed system, the component's performance is sensitive to the temperature difference between them because of the problem of signal skew, and so, the junction temperatures of various components should be kept within a specified range for a high performance system. For most chips, 0.25°C may be the maximum allowable value.

When a thermal analysis was applied to the chip at the component level, the following Fourier's heat conduction equation was usually used

$$q = -K\frac{\partial T}{\partial x}. \qquad (1)$$

However this equation is based on the diffusion mechanism and implies a presumption of infinite thermal propagation speed not applicable for a rapid wave heat transient process. Alternatively, the C–V heat conduction equation, originally proposed from Maxwell equation [1] and then modified by Cattaneo and Vernotte [2–4], can be used for the description of such a rapid heat conduction

$$\tau\frac{\partial q}{\partial t}+q = -K\frac{\partial T}{\partial x} \qquad (2)$$

where τ, defined as $\tau = \alpha/c^2$, is the relaxation time and explained as the build-up period of the commencement of heat flow after a temperature gradient is imposed on the medium; c, the thermal wave propagation speed and α, the thermal diffusivity of the medium.

Comparing the C–V equation (2) with the telegram equation

$$\frac{L}{R}\frac{\partial i}{\partial t}+i = -\frac{1}{R}\frac{\partial E}{\partial x} \qquad (3)$$

we have the complete analogy between heat transfer and electrical current transmission as listed in Table 1. In electrical analogy, the term τ/K in equation (2) is equivalent to the electrical inductance L. The relaxation term can not be neglected in strongly transient process as that the electrical inductance can not be ignored for an rapid alternating current circuit. This is known as thermal wave phenomena.

Several features of high speed IC chip make it important